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Damage detection in discrete vibrating systems

Michele Dilena, Antonino Morassi*

Dipartimento di Georisorse e Territorio, Università degli Studi di Udine, Italy

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Abstract

This paper deals with the identification of a single defect in a discrete spring-mass or beam-like system by measurements of damage-induced shifts in resonance frequencies and antiresonance frequencies. For initially uniform discrete systems, it is shown how the measurement of an appropriate set of frequencies and antiresonances permits unique identification of the damage. The theoretical results are confirmed by comparison with numerical and experimental tests.

1. Introduction

This paper is concerned with the identification of localized damage in discrete vibrating systems, such as spring-mass or beam-like systems, by a minimal number of frequency measurements.

Discrete vibrating systems are quite common structural mechanics. They originate, for example, when finite element models are used to approximate the dynamic behavior of continuous systems (see Ref. [1]). There are other situations, which are important both for the civil and mechanical engineering field, in which real vibrating structures can be accurately modelled as discrete systems. To give few significant examples of this kind, multi-story buildings under seismic base excitation or rotary shafts in turbines can be modelled as discrete spring-mass systems (see Ref. [2]). Moreover, beam-like systems with massless rigid elements connected themselves by elastic joints are largely employed to model the dynamic behavior of mechanical manipulators in robotics.

*Corresponding author.
E-mail addresses: michele_dilena@virgilio.it (M. Dilena), antonino.morassi@uniud.it (A. Morassi).

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Inverse vibration problems for these classes of discrete systems have been extensively investigated since long ago (see Ref. [3] for a complete state-of-the-art and recent results). In most of these studies, researchers focused on finding sufficient conditions to reconstruct uniquely the whole collection of the stiffness and inertia coefficients of the system. As a typical example of this kind of results, it is recalled that a spring-mass system consisting of N masses and stiffnesses can be uniquely reconstruct when two full spectra, corresponding to free–free and fixed–free boundary conditions, are known. Practical situations are quite different since in most cases only the first few frequencies can be determined accurately in experiments and, therefore, data are insufficient to guarantee uniqueness of the identification.

Damage detection problems have some special features. In fact, in most problems of structural diagnostics damage occurs at a limited number of positions and, therefore, when the undamaged system is completely known, only few parameters need to be determined. Such a peculiarity in damage detection has been observed more or less explicitly elsewhere in the literature and has been recently emphasized in Ref. [4].

Most of the available diagnostic techniques for vibrating systems are formulated as an optimality criterion, where the stiffness distribution of a chosen reference configuration of the system is updated so that the first few frequencies closely match the measured ones at a certain level of deterioration, see, for instance, Refs. [5–7]. In a recent paper, Zhu and Wu [8] proposed a method based on the sensitivity analysis for predicting both the locations and magnitude of damage at one or more sites in large mono-coupled periodic systems, using measured changes in the natural frequencies. On the one hand, these optimization techniques have the advantage of allowing for investigation of quite general classes of problems. On the other hand, the lack of satisfactory framework of general properties give rise to several indeterminacy, which, in some cases, may obstruct application to practical problems, see, for example, Refs. [9–11].

Here, following a line of research initiated in Refs. [12,13], and developed subsequently in Refs. [14–16], the damage detection problem in discrete systems, such as spring-mass or beam-like systems, is investigated from a different point of view. The attention is focussed on finding conditions which allow for a rigorous formulation of the damage detection problem from minimal frequency measurements. Despite very extensive literature on inverse problems for discrete systems, both from the mathematical and applied point of view, few results of this kind seem to be available. The present paper deals with the problem of detecting a localized damage in initially uniform discrete spring-mass or beam-like systems. In particular, when a single localized damage is present, a situation simple but quite common in practice, the unknown damage parameters reduce to the stiffness variation and the index corresponding to the damaged spring element. Therefore, it is reasonable to investigate the extent of which the measurement of the damage-induced changes in a pair of frequencies can be useful for identifying the damage.

In order to illustrate the present results, for the sake of simplicity, reference is made to a spring-mass system under free–free boundary conditions. It was found that knowledge of the ratio between the variations of the first two natural frequencies uniquely determines the damage (except for symmetrical positions). Furthermore, the measurement of the variation of the first resonant frequency and the first antiresonance of the point frequency response function corresponding to one end of the spring-mass system, allows for discharging the spurious damage location caused by structural symmetry. As for cantilever or fixed-fixed spring mass systems, it is borne out that by simultaneously employing natural frequencies and antiresonant frequencies, it is possible to significantly reduce the non-uniqueness of the damage location problem.

The proposed identification technique is essentially based on the well-known explicit expression for the damage sensitivity of eigenfrequencies and represents the discrete version of the approach used in Refs. [13,16] to detect damage in continuous beams. Part of the results above are also valid for initially uniform beam-like discrete systems in bending under various set of boundary conditions.

The prediction of the theory and reliability of the diagnostic technique were checked on the basis of results of several pseudo-experimental and experimental dynamic tests performed on damaged systems. Numerical results agree well with analytical predictions.

The plan of the present paper is as follows. Frequency sensitivity to damage and the diagnostic method are illustrated in Section 2. Section 3 is devoted to present theoretical results for initially uniform discrete systems. Numerical applications are discussed in Section 4.

2. Eigenvalue sensitivity to damage and the diagnostic method

2.1. Spring-mass systems

Consider a free–free spring-mass system in an undamaged state consisting of a chain of N masses m_j, $j = 1, \ldots, N$, connected consecutively by linear elastic springs of stiffness k_j, $j = 1, \ldots, N-1$ (see Fig. 1).

The spatial variation of the infinitesimal free vibrations about an equilibrium position of the undamaged system is governed by the discrete eigenvalue problem

$$\mathbf{Ku} = \lambda \mathbf{Mu}, \tag{1}$$

where $\mathbf{u} = (u_1, \ldots, u_N)$, $\mathbf{u} \neq \mathbf{0}$, is the *normal mode* and $\sqrt{\lambda}$ is the associated *natural frequency*. Throughout this paper, the system is assumed to have no material damping. The $N \times N$ symmetrical matrices $\mathbf{K}$ and $\mathbf{M}$ are the stiffness and the inertia, or mass, matrices of the system. $\mathbf{M}$ is diagonal with mass values m_j on the places (j,j), e.g. $\mathbf{M} = \mathrm{diag}(m_1, \ldots, m_N)$. $\mathbf{K}$ is the tridiagonal positive semi-definite matrix given by

$$\mathbf{K} = \begin{pmatrix} k_1 & -k_1 & 0 & \cdots & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & \cdots & 0 & 0 \\ 0 & -k_2 & k_2+k_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{N-2}+k_{N-1} & -k_{N-1} \\ 0 & 0 & 0 & \cdots & -k_{N-1} & k_{N-1} \end{pmatrix}. \tag{2}$$

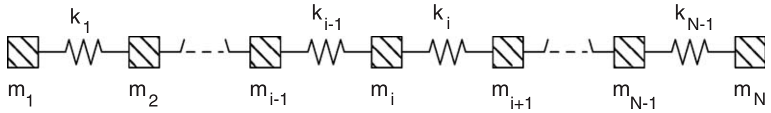

Fig. 1. Spring-mass system.

Eq. (1) has N simple eigenvalues satisfying $0=\lambda_0<\lambda_1<\cdots<\lambda_{N-1}$, and N corresponding eigenvectors $\mathbf{u}^{(j)}$, $j=0,\ldots,N-1$. The eigenvectors $\{\mathbf{u}^{(j)}\}_{j=0}^{N-1}$ form an orthogonal basis for $\mathbb{R}^N$, with respect to the scalar product induced by the mass matrix $\mathbf{M}$, and they can be chosen such that $\mathbf{M}\mathbf{u}^{(j)}\cdot\mathbf{u}^{(k)}=\delta_{jk}$, δ_{jk} being the Kronecker symbol.

Suppose that a structural damage consisting of a reduction Δk_i, $-k_i<\Delta k_i<0$, of the spring stiffness occurs, say, at the ith elastic spring connecting the ith and $(i+1)$th degree of freedom. Therefore, the eigenvalue problem for the damaged system is the following:

$$\widetilde{\mathbf{K}}\widetilde{\mathbf{u}}=\widetilde{\lambda}\mathbf{M}\widetilde{\mathbf{u}}, \tag{3}$$

where

$$\widetilde{\mathbf{K}}-\mathbf{K}=\begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \Delta k_i & -\Delta k_i & \cdots & 0 \\ 0 & \cdots & -\Delta k_i & \Delta k_i & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{matrix} \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} \tag{4}$$

$$\qquad\qquad\qquad i \qquad i+1$$

Eq. (3) has also N real simple eigenvalues satisfying $0=\widetilde{\lambda}_0<\widetilde{\lambda}_1<\cdots<\widetilde{\lambda}_{N-1}$, and N corresponding eigenvectors $\widetilde{\mathbf{u}}^j$, $j=0,1,\ldots,N-1$. The variational formulation of the eigenvalue problem (1) shows that eigenvalues are decreasing functions of k_i, that is

$$0=\widetilde{\lambda}_0=\lambda_0,\quad \widetilde{\lambda}_j\leqslant\lambda_j\quad \text{for } j=1,\ldots,N-1. \tag{5}$$

Moreover, by monotonicity principle, the following interlacing result holds:

$$\lambda_{j-1}\leqslant\widetilde{\lambda}_j\leqslant\lambda_j,\quad j=1,\ldots,N-1. \tag{6}$$

If the damage is small, namely $|\Delta k_i|\ll k_i$, the first-order variation of the eigenvalues with respect to Δk_i may be found by differentiating Eq. (1) with respect to the parameter k_i. Putting

$$\widetilde{\lambda}_j(\widetilde{\mathbf{K}})=\lambda_j(\mathbf{K})+\delta\lambda_j,\quad j=1,\ldots,N-1, \tag{7}$$

and by taking into account the mass-normalization condition of eigenvectors, it can be shown that, neglecting terms of higher order on $|\Delta k_i|$,

$$\delta\lambda_j=\left(\frac{\partial\mathbf{K}}{\partial k_i}\mathbf{u}^{(j)}\cdot\mathbf{u}^{(j)}\right)\Delta k_i, \tag{8}$$

with i and j, $j=1,\ldots,N-1$, fixed indexes, see, for example, Ref. [17]. More explicitly, Eq. (8) gives

$$\delta\lambda_j=\Delta k_i(u_{i+1}^{(j)}-u_i^{(j)})^2. \tag{9}$$

That is, the change in a natural frequency produced by a localized damage may be expressed as the product of two terms, the first of which is proportional to the severity and the second depends only on the location of the damage. This second term is proportional to the square of the *axial* *force* $N_i^{(j)} = k_i(u_{i+1}^{(j)} - u_i^{(j)})$ in the jth mode shape of the undamaged system, between the 2 dof adjacent to the damaged spring, see Ref. [13] for the analogue continuous case.

Eq. (9) has an important consequence; the ratios of the change in two different natural frequencies depend only on the damage location i, not on the damage severity Δk_i, see also Refs. [18,5]. That is, for $\delta\lambda_k < 0$,

$$\frac{\delta\lambda_j}{\delta\lambda_k} = \frac{(u_{i+1}^{(j)} - u_i^{(j)})^2}{(u_{i+1}^{(k)} - u_i^{(k)})^2} \equiv f(i), \tag{10}$$

which can be considered as an equation to be solved with respect to the integer variable i, where i belongs to the set $\{1, \ldots, N-1\}$.

From the practical point of view, once the free vibration problem related to the undamaged system is solved, the behavior of the function $f(i)$ is known and therefore, from a measured value of the ratio $\delta\lambda_j/\delta\lambda_k$, it is possible via numerical methods to estimate the solutions of Eq. (10). For example, one can compute the right-hand side of Eq. (10) for each integer i, $i = 1, \ldots, N-1$, and then find those values of the index which give the value closest to the left-hand side. This procedure gives a good estimation of the possible damage positions which correspond to the measured ratio $\delta\lambda_j/\delta\lambda_k$. In Section 3.1 the diagnostic problem for the simple but very common case of initially uniform spring-mass systems, e.g. systems for which $k_j = k$ and $m_j = m$ for every j, will be considered and studied in detail.

Till now only free–free boundary conditions have been considered. It is worth noticing that the present method can be adapted in such a way as to take general boundary conditions into account. Two important cases will be considered in the following: *cantilever* (C), when the first mass m_1 is fixed ($u_1 = 0$) and *supported* (S), when the first mass and the last Nth mass are fixed ($u_1 = u_N = 0$). For these cases one has

$$(C) \quad \mathbf{M} = \mathrm{diag}(m_2, \ldots, m_N),$$

$$\mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{N-2}+k_{N-1} & -k_{N-1} \\ 0 & 0 & 0 & \cdots & -k_{N-1} & k_{N-1} \end{pmatrix}; \tag{11}$$

$$(S) \quad \mathbf{M} = \mathrm{diag}(m_2, \ldots, m_{N-1}),$$

$$\mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k_{N-3}+k_{N-2} & -k_{N-2} \\ 0 & 0 & 0 & \cdots & -k_{N-2} & k_{N-2}+k_{N-1} \end{pmatrix}. \tag{12}$$

2.2. Beam-like systems

Consider the discrete system consisting of $(N-1)$ masses $\{m_j\}_{j=1}^{N-1}$ linked by massless rigid beam elements, of equal length l, which are themselves connected by $(N-1)$ rotational elastic springs of stiffness $\{k_j\}_{j=1}^{N-1}$. Suppose that the left and the right ends of the whole system are simply supported (see Fig. 2).

The free undamped infinitesimal vibrations of the system are governed by a discrete eigenvalue problem of the kind (1), where the $(N-1)\times(N-1)$ stiffness, $\mathbf{K}$, and inertia, $\mathbf{M}$, symmetrical matrices are given by

$$\mathbf{M} = \mathrm{diag}(m_1, \ldots, m_{N-1}),$$

$$\mathbf{K} = \frac{1}{l^2}\begin{pmatrix} 4k_1+k_2 & -2(k_1+k_2) & k_2 & 0 & \cdots & 0 \\ -2(k_1+k_2) & k_1+4k_2+k_3 & -2(k_2+k_3) & k_3 & \cdots & 0 \\ k_2 & -2(k_2+k_3) & k_2+4k_3+k_4 & -2(k_3+k_4) & \cdots & 0 \\ 0 & k_3 & -2(k_3+k_4) & k_3+4k_4+k_5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & k_{N-2}+4k_{N-1} \end{pmatrix}. \tag{13}$$

Eq. (1), with $\mathbf{M}$ and $\mathbf{K}$ as in Eq. (13), has $(N-1)$ real simple eigenvalues $0<\lambda_1<\cdots<\lambda_{N-1}$ and $(N-1)$ corresponding eigenvectors $\mathbf{u}^{(j)}$, $j=1,\ldots,N-1$, with $\mathbf{M}\mathbf{u}^{(j)}\cdot\mathbf{u}^{(k)}=\delta_{jk}$.

Suppose that a structural damage corresponding to a reduction Δk_i, $-k_i<\Delta k_i<0$, of the rotational spring stiffness occurs, say, at the ith elastic spring connecting the ith and the $(i+1)$th rigid beam element. If the damage is small, on proceeding as it was made in Section 2.1 for the spring-mass system, the first-order variation $\delta\lambda_j$ of the jth eigenvalue with respect to Δk_i can be evaluated from Eq. (8). By considering expression (13) for $\mathbf{K}$, $\delta\lambda_j$ is given by

$$\delta\lambda_j = \Delta k_i \left(\frac{u_{i-1}^{(j)} - 2u_i^{(j)} + u_{i+1}^{(j)}}{l}\right)^2. \tag{14}$$

From Eq. (14), the first-order change $\delta\lambda_j$ in λ_j may be expressed as the product of the severity of the damage, Δk_i, and a second term which is proportional to the square of the bending moment $M_i^{(j)} = k_i((u_{i-1}^{(j)} - 2u_i^{(j)} + u_{i+1}^{(j)})/l)$ in the jth mode shape of the undamaged system, evaluated at the damaged spring, see Ref. [13] for the analogue continuous case.

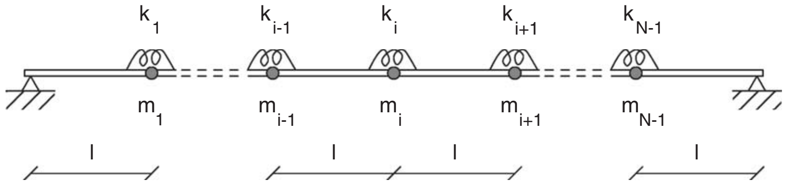

Fig. 2. Simply supported beam-like system.

As before, the ratios of the change in two different natural frequencies depend only on the damage location i, not on the damage severity. For $\delta\lambda_k < 0$ one has

$$\frac{\delta\lambda_j}{\delta\lambda_k} = \frac{(u_{i-1}^{(j)} - 2u_i^{(j)} + u_{i+1}^{(j)})^2}{(u_{i-1}^{(k)} - 2u_i^{(k)} + u_{i+1}^{(k)})^2} \equiv g(i). \tag{15}$$

Eq. (15) can be considered as an equation to be solved, by numerical methods, for example, with respect to the damage position variable i, where i belongs to the set of integer numbers $\{1, \ldots, N-1\}$.

As it was remarked at the end of the previous section, the present method of damage detection can be adapted to consider general boundary conditions. In particular, the free–free boundary conditions case (F–F) will be considered in detail in Section 4. In this case, the ends of the beam shown in Fig. 2 are unconstrained and two point masses m_0, m_N are present at the left end of the first and at the right end of the $(N-1)$th beam element, respectively.

Finally, it should be observed that the assumption of small damage restricts the range of application of the proposed diagnostic method to damaged systems that are a perturbation of the undamaged ones. However, this is not a severe limitation, because in most practical situations it is important to be able to detect damage as soon as it arises.

3. Damage detection in initially uniform discrete systems

In Section 2 it was shown how the problem of detecting a localized damage in a spring-mass or beam-like discrete systems can be formulated on the basis of the knowledge of the damage induced shifts in a pair of natural frequencies. In the present section it will be shown that there are certain cases concerning initially uniform systems, a situation quite common in practice and important for several engineering applications, in which the inverse diagnostic problem can be explicitly solved. In particular, it will be shown how the effects of the non-uniqueness of the solution of the damage location problem may be considerably reduced by means of a careful choice of the frequency data.

3.1. Damage detection in spring-mass systems

In this part, spring-mass discrete systems as those considered in Section 2.1, with uniform inertia and stiffness coefficients in the undamaged configuration, will be investigated, e.g. $m_j = m$ and $k_j = k$ for every value of the index j.

To begin the analysis, the free–free (F) spring-mass system shown in Fig. 1, with a damage at the ith elastic spring, will be considered. The eigenpairs $\{\lambda_j^F, \mathbf{u}^{F(j)}\}_{j=0}^{N-1}$ of the undamaged system can be explicitly evaluated and they are given by

$$\lambda_j^F = 4\frac{k}{m}\sin^2\frac{\vartheta_j^F}{2}, \quad u_i^{F(j)} = \frac{1}{\sqrt{c^{F(j)}}}\cos\frac{\vartheta_j^F}{2}(2i-1), \quad \vartheta_j^F = \frac{j\pi}{N}, \tag{16}$$

$j = 0, 1, \ldots, N-1$, $i = 1, \ldots, N$, where the positive constant $c^{F(j)}$, only depending on j, N and m, has been chosen such that the normalization condition $\mathbf{M}\mathbf{u}^{(j)} \cdot \mathbf{u}^{(j)} = 1$ is satisfied.

By substituting expression (16) of the vibration mode into Eq. (9), after some straightforward computations and a rearrangement of the terms, we can find

$$\delta\lambda_j^F = 4\frac{\Delta k_i}{c^{F(j)}}\sin^2 j\frac{\pi}{2N}\sin^2 j\pi\frac{i}{N}, \tag{17}$$

$j = 0, \dots, N-1$. Note that the fundamental 0th (rigid) mode is always insensitive to damage. By Eq. (10) evaluated for $j = 2$ and $k = 1$, and by using standard trigonometric identities we obtain

$$\frac{\delta\lambda_2^F}{\delta\lambda_1^F} = 4\frac{c^{F(1)}}{c^{F(2)}}\frac{\sin^2\pi/N}{\sin^2\pi/2N}\cos^2\pi\frac{i}{N}, \tag{18}$$

$i = 1, \dots, N-1$. Replacing i/N with the continuous variable s, $s \in (0, 1)$, we can solve explicitly Eq. (18) for the damaged location s and we obtain

$$s_1 = \frac{1}{2\pi}\arccos\left(\frac{1}{2}\frac{c^{F(2)}}{c^{F(1)}}\frac{\sin^2(\pi/2N)}{\sin^2(\pi/N)}\frac{\delta\lambda_2^F}{\delta\lambda_1^F} - 1\right) \tag{19}$$

and the corresponding symmetrical solution

$$s_2 = 1 - s_1. \tag{20}$$

Therefore, the position of the ith damaged spring can be estimated as the integer part of the number obtained by multiplying s_1 or s_2 and N, i.e.

$$i_1 = [s_1 N], \quad i_2 = [(1 - s_1)N]. \tag{21}$$

The above analysis shows that the measurement of the changes in the first and second (elastic) natural frequencies is sufficient to localize the damage, except for symmetrical positions. To complete the identification, the severity of the damage Δk_i can be estimated via Eq. (17).

The non-uniqueness of the damage location due to structural symmetry for the (F) spring-mass system can be eliminated by a combined use of natural frequency and antiresonant frequency measurements. As it is well known, antiresonances correspond to zeros of frequency response functions (frf) $H_{kl}(\sqrt{\lambda})$, where k and l denote the indexes of the response and excitation degree of freedom, respectively. When $k = l$, the zeros of the frf $H_{kk}(\sqrt{\lambda})$ are the natural frequencies of a spring-mass system in which the displacement at the kth degree of freedom is hindered. Therefore, under the assumption of small damage, on proceeding as in Section 2 and with the same notation, the first-order variation of the (square of the) jth antiresonance of the point frf $H_{kk}(\sqrt{\lambda})$ with respect to the damage severity Δk_i may be evaluated by Eq. (9).

Consider now the point frf $H_{11}(\sqrt{\lambda})$. The antiresonances of $H_{11}(\sqrt{\lambda})$ are the (square roots of the) eigenvalues of the spring-mass system (F) with left end, at the first degree of freedom, fixed, namely the eigenvalues λ_j^C of the cantilever system (C) of Eq. (11). It follows that their first-order variation with respect to the damage coincides with the first-order variation $\delta\lambda_j^C$ of the eigenvalues λ_j^C of the cantilever (C). The eigenpairs of the initially uniform (C) system are given by

$$\lambda_j^C = 4\frac{k}{m}\sin^2\frac{\vartheta_j^C}{2}, \quad u_i^{C(j)} = \frac{1}{\sqrt{c^{C(j)}}}\sin\vartheta_j^C(i-1), \quad \vartheta_j^C = \frac{2j-1}{2N-1}\pi, \tag{22}$$

$j=1,\dots,N-1$, $i=2,\dots,N$, where $c^{C(j)}$ is a positive normalization constant depending on j, N and the inertia coefficient m only. By Eq. (9) one has

$$\delta\lambda_j^C = 4\frac{\Delta k_i}{c^{C(j)}}\sin^2\frac{(2j-1)\pi}{2(2N-1)}\cos^2\frac{(2j-1)\pi(2i-1)}{2(2N-1)}, \tag{23}$$

$j=1,\dots,N-1$, and, therefore, taking $\delta\lambda_j=\delta\lambda_1^F$ and $\delta\lambda_k=\delta\lambda_1^C$ in Eq. (10), after some easy calculations, it follows that

$$\frac{\delta\lambda_1^F}{\delta\lambda_1^C} = \frac{c^{C(1)}}{c^{F(1)}}\frac{\sin^2\pi/2N}{\sin^2\pi/2(2N-1)}\frac{\sin^2\pi i/N}{\cos^2(\pi/2)(2i-1)/(2N-1)}, \tag{24}$$

$i=2,\dots,N$. The function $h(i)=\sin^2\pi i/N/\cos^2(\pi/2)(2i-1)/(2N-1)$ of the integer variable i is strictly increasing in the interval $2\leqslant i\leqslant N$. By proceeding as exemplified above in solving Eq. (18), Eq. (24) can be uniquely inverted to obtain an estimate of the damage location. Therefore, it turns out that from the knowledge of the first elastic natural frequency under boundary conditions (F) and of the first antiresonant frequency of the point frf $H_{11}(\sqrt{\lambda})$ it is possible to localize uniquely the damage.

The present technique can be also adapted to analyze the damage identification in a spring-mass system under (C) boundary conditions. This case is rather important for applications, because it describes the dynamic behavior of shear-type buildings under seismic base excitation. In particular, it will be now examined to what extent the measurement of the first resonant frequency of the (C) system and the first antiresonance of the point frf $H_{NN}(\sqrt{\lambda})$ measured at the free upper end is useful to localize the damage. It is worth noticing that this situation describes the dynamic test in which an horizontal force is applied to the top floor and, at the same level, the dynamic response is measured so to determine the point frf $H_{NN}(\sqrt{\lambda})$. Note that the required data for damage identification are extracted only from a single frf measurement.

The antiresonances of the point frf $H_{NN}(\sqrt{\lambda})$ are the (square roots of the) eigenvalues λ_j^S of the supported spring-mass system (S) of Eq. (12). The normalized eigenpairs of the initially uniform (S) system have the expression

$$\lambda_j^S = 4\frac{k}{m}\sin^2\frac{\vartheta_j^S}{2},\quad u_i^{S(j)} = \frac{1}{\sqrt{c^{S(j)}}}\sin\vartheta_j^S(i-1),\quad \vartheta_j^S=\frac{j\pi}{N-1}, \tag{25}$$

$j=1,\dots,N-2$, $i=2,\dots,N-1$, where $c^{S(j)}$ is a positive normalization constant depending on j, N and the inertia coefficient m only. By substituting the expression of the fundamental vibrating modes of the (C) and (S) systems in Eq. (10), after some calculations, we can find

$$\frac{\delta\lambda_1^S}{\delta\lambda_1^C} = c(N)\frac{\cos^2\pi(2i-1)/(2(N-1))}{\cos^2(\pi/2)(2i-1)/(2N-1)}, \tag{26}$$

where the unknown integer variable i ranges from 2 to $(N-1)$ and the known constant $c(N)$ depends on N only. Replacing the discrete position variable $(2i-1)/(2N-1)$ with the continuous variable s, $s\in(0,1)$, we can show that the function $f(s)=\cos^2\pi s/\cos^2(\pi s/2)$ has the behavior illustrated in Fig. 3.

Therefore, if $\delta\lambda_1^S/(c(N)\delta\lambda_1^C)>1$ there is a unique solution of diagnostic problem and the damage location is placed near the free end, e.g. in the interval $s\in(s^*,1)$, where $s^*=\frac{2}{3}$. If

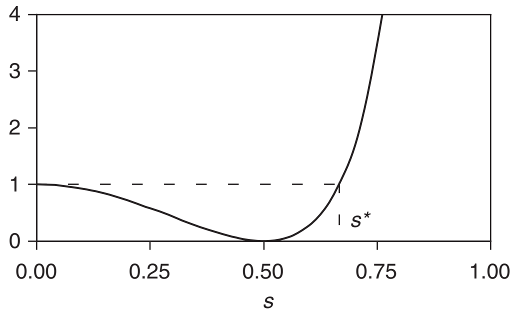

Fig. 3. Graph of the function $f(s) = \cos^2\pi s/\cos^2\pi s/2$ in the interval [0, 1).

$0 < \delta\lambda_1^S/(c(N)\delta\lambda_1^C) \leqslant 1$ there are two distinct damage locations which correspond to the same ratio between the first natural frequency and the first antiresonant frequency, one located in the left part of the system, adjacent to the fixed end, the other located in the interval $(\frac{1}{2}, s^*)$. Finally, if $\delta\lambda_1^S = 0$, the damage is located at the mid-point of the system.

3.2. Damage detection in beam-like systems

The discrete model, which will be mainly investigated, is a beam-like system under supported-supported boundary conditions as that shown in Fig. 2. Suppose that the undamaged system is formed by N rigid beam elements of equal length l and has uniform inertia and stiffness coefficients, e.g. $m_j = m$ and $k_j = k$ for every value of the index j. The normalized eigenpairs of the system can be explicitly determined and they are given by

$$\lambda_j^{S-S} = 16\frac{k}{ml^2}\sin^4\frac{\vartheta_j^S}{2}, \quad u_i^{S-S(j)} = \frac{1}{\sqrt{c^{S-S(j)}}}\sin\vartheta_j^{S-S}i, \quad \vartheta_j^{S-S} = \frac{j\pi}{N}, \tag{27}$$

$j = 1, \ldots, N-1$, $i = 1, \ldots, N-1$, with $c^{S-S(j)} > 0$ a suitable normalization constant.

Suppose that a damage occurs at the ith rotational spring connecting the ith and the $(i+1)$th rigid beam element. The structural damage corresponds to a reduction Δk_i, $-k_i < \Delta k_i < 0$, of the spring stiffness. Assuming the damage small, that is $|\Delta k_i| \ll k_i$, the first-order variation $\delta\lambda_j^{S-S}$ of the jth eigenvalue with respect to Δk_i can be evaluated from Eq. (14). By substituting expression (27) of the vibration mode into Eq. (14), after some straightforward calculations, we find that the ratio (15) between the first-order variations in the second and first eigenvalue is given by

$$\frac{\delta\lambda_2^{S-S}}{\delta\lambda_1^{S-S}} = c(N)\cos^2\pi\frac{i}{N}, \tag{28}$$

$i = 1, \ldots, N$, where the known constant $c(N) > 0$ only depends on N. Replacing i/N with the continuous variable s, $s \in (0, 1)$, we can solve explicitly Eq. (28) for the damage location s and we obtain

$$s_1 = \frac{1}{\pi}\arccos\left(\frac{2}{c(N)}\frac{\delta\lambda_2^{S-S}}{\delta\lambda_1^{S-S}} - 1\right) \tag{29}$$

and the corresponding symmetrical position

$$s_2 = 1 - s_1. \quad (30)$$

The position of the damaged spring can be evaluated as before, by taking the integer part of the number obtained by multiplying s_1 and s_2 times N, i.e. $i_1 = [s_1 N]$ and $i_2 = [(1 - s_1)N]$. Therefore, the above analysis shows that the measurement of the changes in the first and second natural frequencies of a (S–S) beam-like system is sufficient to localize the damage, except for symmetrical positions. Damage identification results for different set of boundary conditions will be considered in the next section.

4. Applications

In previous section it was shown how measurements of natural frequencies and antiresonant frequencies may be used to assess the location as well as the severity of a localized damage in a spring-mass system (Section 3.1) or in a beam-like system (Section 3.2). The present section is devoted to illustrate some applications of numerical and experimental character. The proposed diagnostic technique has been tested on several discrete systems and under different damage scenarios. In particular, the following analysis will concern with a rotating shaft carrying six disks (Example 1), a shear-type building (Example 2), a spring-mass system (Example 3) and a beam-like system (Example 4). In the last two examples, discrete vibrating systems are used as finite element approximation of the longitudinal and transversal vibrations of a continuous beam with a single crack of increasing depth. In Examples 1 and 2, the inverse problem of damage detection is solved using pseudo-experimental (simulated) data, that is, the frequencies are obtained by solving numerically the direct problem in undamaged condition and in some damage state defined by two damage parameters i, the damage location, and Δk_i, the damage severity. In the remaining two cases, Examples 3 and 4, the diagnostic procedure is tested on the basis of real experimental data coming from some dynamic tests performed on steel cracked beams.

4.1. Example 1

The system consists of six equal disks, with mass polar moment of inertia $J_0 = 0.053\,\mathrm{kg\,m^2}$, connected by five equal massless shafts of length $L_0 = 0.4\,\mathrm{m}$ and with torsional stiffness $GJ = 46\,\mathrm{kN\,m^2}$. The system is supported at both ends by means of frictionless sleeves in such a way that the entire system can rotate freely as a whole (see Fig. 4).

One main case of damage among several studied is presented and discussed in detail: it is illustrative of the main feature of the inverse problem and of the identification technique. This case is characterized by a damage in the second shaft or, equivalently, in the torsional spring k_2 connecting the second and the third disk (from the left hand side in Fig. 4). Three levels of damage $D1$, $D2$, $D3$ were considered, corresponding to a reduction of the 5%, 10% and 20% of the initial value of the torsional stiffness k_2, respectively for $D1$, $D2$, $D3$. The corresponding average variations of the frequencies are about 0.6%, 1.3% and 3% for $D1$, $D2$, $D3$, respectively (see Table 1).

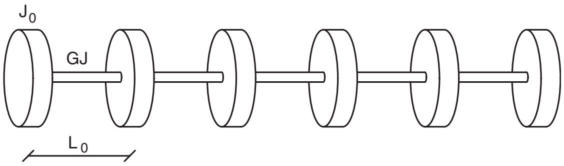

Fig. 4. Rotating shaft with disks.

Table 1
Analytical frequencies and antiresonances (of the point frf $H_{11}(\sqrt{\lambda})$) of the system of Fig. 4

Mode number	Undamaged f_n^{undam}	Damage $D1$		Damage $D2$		Damage $D3$	
		f_n^{dam}	$\Delta f_n\%$	f_n^{dam}	$\Delta f_n\%$	f_n^{dam}	$\Delta f_n\%$
Frequencies							
0	0.00	0.00	—	0.00	—	0.00	—
1	121.29	120.49	0.7	119.62	1.4	117.56	3.1
2	234.31	232.80	0.6	231.17	1.3	227.56	2.9
3	331.37	331.37	0.0	331.37	0.0	331.37	0.0
4	405.84	403.15	0.7	400.17	1.4	393.40	3.1
5	452.66	449.98	0.6	447.61	1.1	443.72	2.0
Antiresonances							
1	66.69	66.17	0.8	65.60	1.6	64.30	3.6
2	194.67	194.53	0.1	194.37	0.2	193.99	0.3
3	306.88	306.01	0.3	305.02	0.6	302.55	1.4
4	394.23	390.59	0.9	386.69	1.9	378.35	4.0
5	449.64	447.45	0.5	445.53	0.9	442.39	1.6

Undamaged configuration: $k = GJ/L_0 = 115\,\text{kN m}$, $J_0 = 0.053\,\text{kg m}^2$. Damage scenarios: $k_2^{\text{dam}}/k_2^{\text{undam}} = 0.95, 0.90, 0.80$ for damages $D1$, $D2$, $D3$, respectively. Frequency values f_n in Hz. $\Delta f_n\% = 100(f_n^{\text{undam}} - f_n^{\text{dam}})/f_n^{\text{undam}}$.

The frequency values of the undamaged shaft and their values associated with the cases of damage are reported in Table 1; the latter have been obtained by solving the perturbed eigenvalue problem (3). The results of identification are presented in Tables 2 and 3. It is possible to observe that, in the absence of errors on the data, the pair of two solutions (21) predicted by the theory for the mathematical problem when the first two (elastic) frequencies are used, contains the real solution of the damage problem (damage in the spring number 2) and its symmetrical counterpart (damage in the spring number 4). The deviations from the effective severity of the damage are negligible for all the cases considered. As it was shown in Section 3.1, the use of the first resonance and of the first antiresonance of the frf corresponding to one end of the system, allows for a unique localization of the damage. This theoretical result is confirmed by the numerical applications (see Table 2).

The damage analysis have been developed in absence of errors so far, but, as it is well known, the results of most identification techniques strictly depend on possible measurement and

Table 2
Determination of the damage location by using the first two frequencies (Eq. (18)) and first frequency-first antiresonance of the point frf $H_{11}(\sqrt{\lambda})$ (Eq. (24))

Data	Damage $D1$		Damage $D2$		Damage $D3$	
	s_1N	s_2N	s_1N	s_2N	s_1N	s_2N
Analysis without errors						
Resonances	2.008	3.993	2.016	3.984	2.035	3.965
Res.–antires.	2.010		2.020		2.043	
Analysis with errors—case (a)						
Resonances	2.030	3.970	2.025	3.975	2.042	3.958
Res.–antires.	No solution		No solution		No solution	
Analysis with errors—case (b)						
Resonances	1.987	4.013	2.008	3.992	2.029	3.971
Res.–antires.	No solution		No solution		No solution	

Actual damage location: spring number 2. Damage scenarios $D1$, $D2$, $D3$ as in Table 1. Errors on frequencies f_n and antiresonances f_n^a: case (a) err($n = 1$) = 2%, err($n = 2$) = 2%; case (b) err($n = 1$) = 2%, err($n = 2$) = 4%. Err(n) = 1% means that the frequency value used in the identification procedure is $f_n^{\text{exact}}(1 + 0.01)$.

Table 3
Determination of the damage severity by using the first two frequencies (Eq. (18)) and the first frequency-first antiresonance of the point frf $H_{11}(\sqrt{\lambda})$ (Eq. (24))

Data	Damage $D1$		Damage $D2$		Damage $D3$	
	s_1N	s_2N	s_1N	s_2N	s_1N	s_2N
Analysis without errors						
Resonances	0.95	0.95	0.89	0.89	0.76	0.76
Res.–antires.	0.95		0.89		0.76	
Analysis with errors—case (a)						
Resonances	0.99	0.99	0.93	0.93	0.80	0.80
Res.–antires.	No solution		No solution		No solution	
Analysis with errors—case (b)						
Resonances	0.99	0.99	0.93	0.93	0.80	0.80
Res.–antires.	No solution		No solution		No solution	

Actual damage severity: $k_2^{\text{dam}}/k_2^{\text{undam}} = 0.95, 0.90, 0.80$ for damages $D1$, $D2$, $D3$, respectively. Errors on frequencies f_n and antiresonances f_n^a: case (a) err($n = 1$) = 2%, err($n = 2$) = 2%; case (b) err($n = 1$) = 2%, err($n = 2$) = 4%. Err(n) = 1% means that the frequency value used in the identification procedure is $f_n^{\text{exact}}(1 + 0.01)$.

modelling errors. To take the effect of errors in the experimental data into account, the previous cases in which the data were corrupted by some noise have been considered (see Tables 2 and 3). In general, it is possible to observe that, in the inverse problem solution, the damage detection procedure is quite stable when resonances are used as data in identification. Conversely, the inverse problem has no solution when antiresonance data are corrupted by some noise. The above results suggest that the noise errors on antiresonances are amplified strongly with respect to cases in which frequency data are used in identification. This is in agreement with the recent results found in Ref. [16] concerning crack detection in continuous beams.

4.2. Example 2

The second example concerns a spring-mass system which models the vibrations of a shear-type building. The proposed damage detection technique has been tested on several systems with different number of floors. Here, for the sake of brevity, a fifth-story shear-type building will be mainly investigated. The column masses are considered negligible in comparison with the floor masses, which have been assumed equal to $m = 6 \times 10^5$ kg. The shear stiffness of each story of the undamaged system is equal to $k = 2 \times 10^9$ N/m.

To apply the proposed procedure of damage detection, one damage scenario among several studied will be analyzed in detail, e.g. a damage located at the first story of the building, and three damage configurations corresponding to a reduction of 5% ($D1$), 10% ($D2$) and 20% ($D3$) of the initial value of shear stiffness k_1 will be considered. The corresponding frequency and antiresonance (of the point frf of the last floor) variations are shown in Table 4. The results of

Table 4
Analytical frequencies and antiresonances (of the point frf $H_{NN}(\sqrt{\lambda})$) of the shear-type building

Mode number	Undam.	Damage $D1$		Damage $D2$		Damage $D3$	
	f_n^{undam}	f_n^{dam}	$\Delta f_n\%$	f_n^{dam}	$\Delta f_n\%$	f_n^{dam}	$\Delta f_n\%$
Frequencies							
1	2.62	2.59	0.9	2.56	1.9	2.50	4.2
2	7.63	7.58	0.8	7.51	1.6	7.38	3.4
3	12.03	11.97	0.5	11.91	1.1	11.78	2.1
4	15.46	15.42	0.3	15.38	0.5	15.31	1.0
5	17.63	17.62	0.1	17.61	0.1	17.59	0.3
Antiresonances							
1	5.68	5.63	0.9	5.57	1.9	5.44	4.2
2	10.80	10.73	0.7	10.66	1.3	10.50	2.8
3	14.87	14.82	0.3	14.77	0.7	14.67	1.3
4	17.48	17.46	0.1	17.45	0.2	17.42	0.3

Undamaged configuration: story shear stiffness $k = 2 \times 10^9$ N/m, floor mass $m = 6 \times 10^5$ kg. Damage scenarios: $k_1^{\text{dam}}/k_1^{\text{undam}} = 0.95, 0.90, 0.80$ for damages $D1$, $D2$, $D3$, respectively. Frequency values f_n in Hz. $\Delta f_n\% = 100(f_n^{\text{undam}} - f_n^{\text{dam}})/f_n^{\text{undam}}$.

Table 5
Determination of the damage location by using the first two frequencies and first frequency-first antiresonance of the point frf $H_{NN}(\sqrt{\lambda})$ (Eq. (26))

Data	Damage $D1$		Damage $D2$		Damage $D3$	
	s_1N	s_2N	s_1N	s_2N	s_1N	s_2N
Analysis without errors						
Resonances	1.016	3.174	1.034	3.169	1.071	3.158
Res.–antires.	1.021	3.906	1.042	3.903	1.089	3.897
Analysis with errors—case (a)						
Resonances	1.002	3.178	1.020	3.173	1.059	3.162
Res.–antires.	0.991	3.910	1.014	3.907	1.063	3.901
Analysis with errors—case (b)						
Resonances	0.959	3.190	0.978	3.185	1.021	3.173
Res.–antires.	0.887	3.920	0.917	3.918	0.977	3.911

Actual damage location: spring number 1. Damage scenarios $D1$, $D2$, $D3$ as in Table 4. Errors on frequencies f_n and antiresonances f_n^a: case (a) err($n=1$) = 0.5%, err($n=2$) = 1%; case (b) err($n=1$) = 1%, err($n=2$) = 3%. Err(n) = 1% means that the frequency value used in the identification procedure is $f_n^{\text{exact}}(1+0.01)$.

identification are presented in Tables 5 and 6. For the sake of completeness, the results of identification using variations in the first two frequencies have been also included in Tables 5 and 6. It is possible to note that the predictions of the theory for the mathematical problem are confirmed. In fact, the damage location problem has two different solutions corresponding to the first and third floor, the first and fourth floor, respectively when the first two frequencies and the first frequency and first antiresonance are used as data (see Fig. 3).

The deviations from the exact severity of the damage are negligible for the cases considered. Finally, as in the previous example, errors in the data are amplified strongly when antiresonance measurements are used in identification.

4.3. Example 3

In this example the diagnostic technique is applied to spring-mass systems which represent a discretization of continuous axially vibrating bars. The discrete system is regarded as consisting of N lumped rigid masses $m = \rho\Delta L$ concentrated at N points equally distributed along the beam axis, where ρ is the linear mass density of the bar and $\Delta L = L/(N-1)$ is the distance between two consecutive masses. The segment of bar between the lumped masses is assumed to be massless and is modelled as a linear elastic spring of stiffness $K = EA/\Delta L$, where EA is the axial stiffness of the rod. The experimental models consisted of uniform steel bars under free–free boundary conditions. Every specimen was damaged by saw-cutting the transversal cross-section. The width of each crack was equal to 1.5 mm and, because of the small level of the excitation, during the dynamic tests each crack remains always open. The proposed diagnostic procedure was tested on

Table 6
Determination of the damage severity by using the first two frequencies and first frequency-first antiresonance of the point frf $H_{NN}(\sqrt{\lambda})$ (Eq. (26))

Data	Damage $D1$		Damage $D2$		Damage $D3$	
	s_1N	s_2N	s_1N	s_2N	s_1N	s_2N
Analysis without errors						
Resonances	0.95	0.90	0.89	0.80	0.77	0.57
Res.–antires.	0.95	0.84	0.89	0.67	0.77	0.29
Analysis with errors—case (a)						
Resonances	0.95	0.90	0.89	0.80	0.77	0.57
Res.–antires.	0.95	0.84	0.89	0.67	0.77	0.29
Analysis with errors—case (b)						
Resonances	0.95	0.90	0.89	0.80	0.77	0.56
Res.–antires.	0.95	0.84	0.89	0.66	0.77	0.28

Actual damage severity: $k_1^{\text{dam}}/k_1^{\text{undam}} = 0.95, 0.90, 0.80$ for damages $D1$, $D2$, $D3$, respectively. Errors on frequencies f_n and antiresonances f_n^a: case (a) $\text{err}(n = 1) = 0.5\%$, $\text{err}(n = 2) = 1\%$; case (b) $\text{err}(n = 1) = 1\%$, $\text{err}(n = 2) = 3\%$. $\text{Err}(n) = 1\%$ means that the frequency value used in the identification procedure is $f_n^{\text{exact}}(1 + 0.01)$.

several cracked steel beams, with a double T cross-section or with solid cross-section. In particular, the following analysis will concern with the steel rod of square solid cross-section shown in Fig. 5. The results obtained for this specimen are representative of the experimental and theoretical questions arisen in the course of damage identification.

By using an impulsive dynamic technique, the first lower natural frequencies of the undamaged bar and of the bar under a series of three damage configurations ($D1$, $D2$ and $D3$, see Fig. 5) were determined. The rod was suspended by two steel wire ropes to simulate free–free boundary conditions. The excitation was introduced at one end by means of an impulse force hammer, while the axial response was measured by a piezoelectric accelerometer fixed in the center of an end cross-section of the rod. Vibration signals were acquired by a dynamic analyzer and then worked out in the frequency domain to measure the relevant frequency response term (inertance). The well-separated vibration modes and the very small damping allowed us to identify the natural frequencies by means of the single mode technique, see Ref. [19] (Section 5, Second Experiment) for a complete account of the experiment. The damage configurations were obtained by introducing a notch of increasing depth at 1.00 m from one end. Table 7 compares the experimental natural frequencies and their corresponding analytical estimates for the continuous and discrete model of the undamaged rod. The discrete model of the rod has $N = 40$ dof and the real damage is located in correspondence of the 14 spring, between the 14 and 15 dof. The analytical model turns out to be extremely accurate for all the configurations under investigation and the percentage discrepancy between the measured and the analytical values of the first lower natural frequencies is less than 0.5%. The severity and the location of the damage have been achieved by applying formulas (19) and (20).

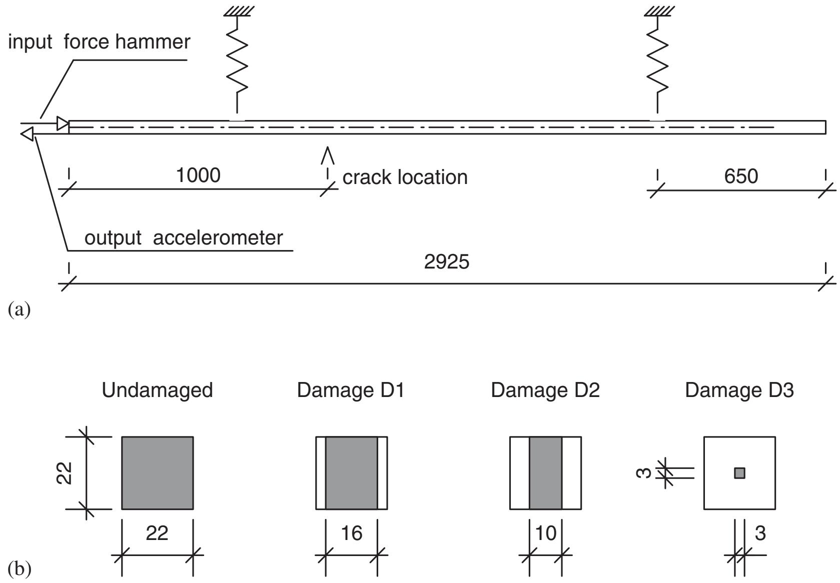

Fig. 5. Free–free axially vibrating beam with a single crack: damaged configurations and discrete model. Lengths in mm.

Table 7
First lower experimental frequencies of the rod

Mode number	Undamaged		Damage $D1$	Damage $D2$	Damage $D3$
	Exper.	40 dof system			
1	882.25	882.00	881.50	879.30	831.00
2	1764.60	1762.70	1763.30	1759.00	1679.50
3	2645.80	2640.60	2644.00	2647.00	2646.50
4	3530.30	3514.50	3526.80	3516.50	3306.00

Undamaged configuration: axial stiffness $EA = 9.95 \times 10^7$ N, linear mass density $\rho = 3.735$ kg/m, length $L = 2.925$ m; discrete model: $N = 40$ dof, $m = 0.273$ kg, $k = 1.363 \times 10^9$ N/m. Crack location: $s = 1.000$ m from the left end. Damage scenarios: $D1$, $D2$, $D3$, see Fig. 4. Frequency values in Hz.

The results of the identification are summed up in Table 8. For the sake of completeness, the estimated interval of possible location of the crack is included. With reference to the localization of the cracked cross-section, the accuracy of the method proves to be satisfactory in all the cases tested. Concerning the estimation of the damage severity, the diagnostic procedure gives increasing values of the stiffness reduction for damage configuration $D1$, $D2$ and $D3$, respectively.

4.4. Example 4

This last example concerns with beam-like discrete systems which are used to discretize continuous beams in bending vibrations. In particular, attention is focussed on a continuous steel

Table 8
Damage identification by using the first two frequencies (Eqs. (19) and (20)) for the rod of Fig. 5. Actual crack location: $s = 1.000$ m from the left end

Damage	Identified damaged spring number	Damage location (m)		Damage severity $\frac{k^{\text{undam}} - k^{\text{dam}}}{k^{\text{undam}}} 100$
		From	To	
*D*1	14	0.951	1.024	98.9
	26	1.828	1.901	
*D*2	14	0.951	1.024	95.6
	26	1.828	1.901	
*D*3	14	0.951	1.024	29.0
	26	1.828	1.901	

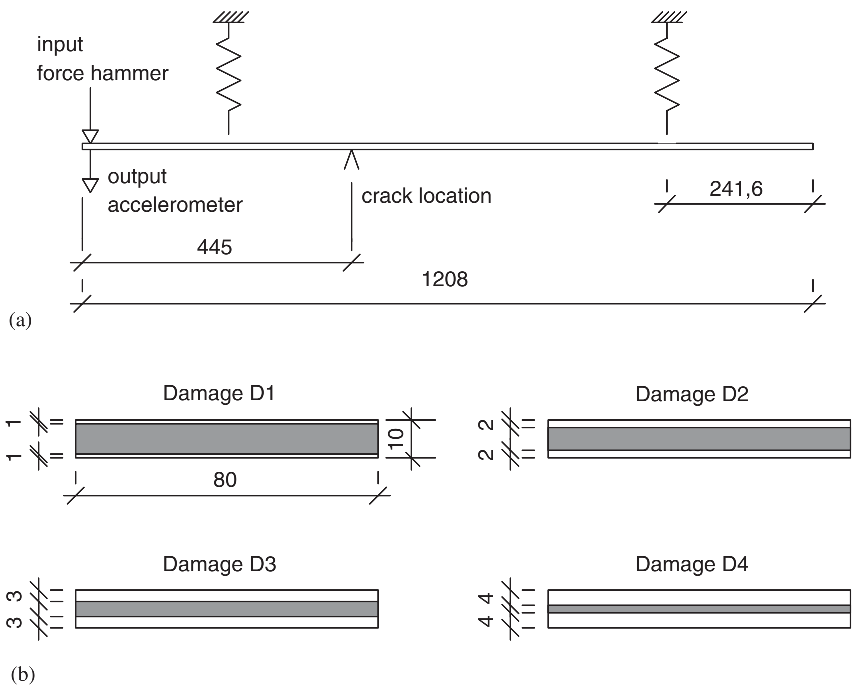

Fig. 6. Free–free bending vibrating beam with a single crack: damaged configurations and discrete model. Lengths in mm.

beam, with rectangular solid cross-section, under free–free boundary conditions (see Fig. 6). By adopting an experimental technique similar to that of Example 3, the undamaged beam and four damaged configurations *D*1–*D*4 were studied, see Ref. [16] for more details on the dynamic experiments. Damage was obtained by introducing a symmetrical saw-cut of depth 1, 2, 3, 4 mm for damage configurations $D1$–$D4$, respectively, at the cross-section 0.445 m far from the left end. The beam was excited transversally at the left end by means of a force hammer and the transversal response at the same end was acquired by a piezoelectric accelerometer. Table 9 shows the measured and analytical values for the first four lower modes of the continuous model of the beam. The discrete model was obtained by concentrating the mass of the continuous beam in $N = 50$ points equally distributed along the beam axis, e.g. $m = \rho \Delta L$, where ρ is the linear mass density of the bar and $\Delta L = L/N$ is the distance between two consecutive masses. The segment of bar between the lumped masses is assumed to be rigid and massless. The elastic compliance of the continuous beam is lumped at the joints connecting two consecutive segments and it is modelled as a linear elastic rotational spring of stiffness $K = EI/\Delta L$, where EI is the bending stiffness of the beam. The discrete model is very accurate in the frequency range considered (see Table 9).

Table 9
First lower experimental frequencies of the beam

Mode number	Undamaged		Damage $D1$	Damage $D2$	Damage $D3$	Damage $D4$
	Exper.	50 dof system				
1	36.60	37.36	36.50	36.20	35.40	32.20
2	100.80	102.89	100.70	100.10	98.40	92.90
3	197.70	201.39	197.70	197.60	197.50	197.30
4	326.80	332.16	326.40	324.00	317.90	298.80

Undamaged configuration: bending stiffness $EI = 1477\,\mathrm{N\,m^2}$, linear mass density $\rho = 6.3\,\mathrm{kg/m}$, length $L = 1.208\,\mathrm{m}$; discrete model: $N = 50$ dof, $m = 0.1522\,\mathrm{kg}$, $k = 61134\,\mathrm{N/m}$. Crack location: $s = 0.445\,\mathrm{m}$ from the left end. Damage scenarios: $D1$, $D2$, $D3$, $D4$, see Fig. 6. Frequency values in Hz.

Table 10
Damage identification by using the first two frequencies

Damage	Identified damaged spring number	Damage location (m)		Damage severity $\frac{k^{\mathrm{undam}} - k^{\mathrm{dam}}}{k^{\mathrm{undam}}} 100$
		From	To	
$D1$	20	0.459	0.483	98.6
	29	0.676	0.701	
$D2$	19	0.435	0.459	94.0
	30	0.701	0.725	
$D3$	18	0.411	0.435	81.5
	31	0.725	0.749	
$D4$	19	0.435	0.459	36.8
	30	0.701	0.725	

Actual crack location: $s = 0.445\,\mathrm{m}$ from the left end.

The results of identification are reported in Table 10. The crack location in the discrete model is determined using the variations of the first and second frequencies and solving numerically with respect to the damage location an equation analogous to Eq. (16). The analysis shows that the above data is sufficient to localize the damage, except for symmetrical positions. Concerning the determination of crack location, the results are in good agreement with theoretical expectations for all the damage scenarios considered. The estimates on the damage severity confirm that damage progressively increasing from level $D1$ to level $D4$.

5. Conclusions

This paper is concerned with the identification of a single defect in a spring-mass or beam-like discrete system from a knowledge of the damage-induced changes in resonance frequencies and antiresonance frequencies. In the case of initially uniform discrete systems, it is shown how an appropriate use of frequencies and antiresonances may be useful for the unique identification of the damage. Numerical results are in good agreement with the theory when analytical or experimental data are employed in identification.

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